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On Structured Tangential Nevanlinna-Pick Interpolation

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Abstract

This paper is a continuation of the work of Bercovici-Foias-Tannenbaum [BFT89a, BFT89b, BFT90] on interpolation problems were the interpolating functions are bounded not in norm but in spectral radius or structured singular values. These problems arise naturally in the robust control of systems with structured uncertainty. Here we extend our structured matrix Nevanlinna-Pick interpolation theory to the (one-sided) tangential case. The complete structured (two-sided) tangential interpolation theory will be presented elsewhere. With these results, we have laid the groundwork for a rigorous analytic procedure for μ-synthesis.

1 Introduction

In our previous papers [BFT89b, BFT90], we proved a spectral and structured generalization of the commutant lifting theorem which allowed us to extend the standard matrix Nevanlinna–Pick interpolation theory to new problems where the interpolating functions are bounded in spectral radius or structured singular value, respectively. Motivated by a number of problems in control engineering, we further extended our results on matrix spectral interpolation to the tangential case [BFT89b]. Here we take a step further and extend the matrix structured Nevanlinna–Pick interpolation to the tangential case.

In order to describe our results, let us briefly consider the standard Nevanlinna–Pick theory in the matrix case. For $\alpha_i \in \mathbf{D}$ (the open unit disk) distinct, $1 \leq i \leq n$, let $F_1, \ldots, F_n$ be $N \times N$ complex matrices. Then we are interested in finding necessary and sufficient conditions for the existence of an analytic (in $\mathbf{D}$) matrix-valued function $F(z)$ with $F(\alpha_i) = F_i, 1 \leq i \leq n$, and such that $\|F\|_\infty \leq 1$. It is well known that the existence of F can be reduced to the determination of the positivity of certain Hermitian *Nevanlinna-Pick matrix*. (This fact can be deduced, e.g., from the commutant lifting theorem [SNF68, SNF70].) In [BFT89a] we studied the problem of bounding the spectral radius of the interpolating functions giving necessary and sufficient conditions for the existence of an interpolating function F with spectral radius bounded away from 1. This was derived as a consequence of a spectral commutant lifting result. Similar results were obtained later for interpolating functions F with structured singular value bounded by 1.

A number of important control problems lead naturally to tangential interpolation problems. The simplest of these can be formulated as follows: let $a_i, a_i' \in \mathbf{C}^N$ be non-zero vectors. We want necessary and sufficient conditions for the existence of an analytic (in $\mathbf{D}$) $N \times N$ matrix-valued function F such that $F(\alpha_i)a_i = a_i'$ and such that $\|F\| \leq 1$. This is the *standard tangential Nevanlinna-Pick* interpolation problem. Fedčina [Fed75] showed that this question reduces to the determination of the positivity of certain Hermitian matrix. Using a commutant lifting approach we showed in [BFT89b] that the spectral version of the above problem, e.g. $\|F\|_\rho \leq 1$, has a solution if an only if a *spectral Nevanlinna-Pick matrix* is positive. In this note we solve a variation of the above problem in which the structured singular value of F is bounded away from 1, e.g. $\|F\|_\mu \leq 1$. As in [BFT89b] this will be deduced as a consequence of a commutant lifting theorem.

2 Preliminaries

Throughout this paper, by "Hilbert space" we will mean "complex separable Hilbert space". By "operator" we shall always mean "bounded linear operator," unless explicitly stated otherwise. All the Hardy spaces are defined in the unit disk, $\mathbf{D}$. $H^\infty(\mathbf{C}^{N\times M})$ is the set of $N \times M$ matrix valued functions essentially bounded on the unit circle, with analytic extension to $\mathbf{D}$. For $\mathcal{E}$ be a finite dimensional Hilbert space, $H^2(\mathcal{E}) := H^2 \otimes \mathcal{E}$ is the Hilbert space of square-summable $\mathcal{E}$-valued power series. For $\mathcal{H}$ a Hilbert space, $\mathcal{L}(\mathcal{H})$ denotes the set of operators on $\mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H})$, $\{T\}'$ denotes the commutant of T, that is $\{T\}' := \{A \in \mathcal{L}(\mathcal{H}) : AT = TA\}$.

Given a bounded analytic function $F : \mathbf{D} \to \mathcal{L}(\mathcal{E})$, we can consider the multiplication operator $M_F : H^2(\mathcal{E}) \to H^2(\mathcal{E})$ defined by $(M_F f)(z) := F(z)f(z), F \in H^2(\mathcal{E}), z \in \mathbf{D}$. M_F commutes with the unilateral shift S given by $(Sf)(z) := zf(z), F \in H^2(\mathcal{E})$, $z \in \mathbf{D}$ and satisfies the norm equality $\|M_F\| = \|F\|_\infty$.

Let $\Delta \subset \mathcal{L}(\mathcal{E})$ be a block diagonal algebra. For $A \in \mathcal{L}(\mathcal{E})$, $A \neq 0$, we define the *structured singular value*

$$\mu_\Delta(A) \;:=\; [\inf\{\|X\| : X \in \Delta, -1 \in \sigma(AX)\}]^{-1} \qquad (1)$$

Moreover we set

$$\hat{\mu}_\Delta(A) \;:=\; \inf\{\|DAD^{-1}\| : \ D \text{ invertible}, D \in \Delta'\} \qquad (2)$$

where Δ' is the commutant of Δ, (i.e., the set of all operators commuting with Δ.)

Remark 2.1 It is well known that $\mu_\Delta(A) \leq \hat{\mu}_\Delta(A)$. In practice it is $\hat{\mu}_\Delta(\cdot)$ that is used for control problems. Note also that $\rho(A) \leq \mu_\Delta(A) \leq \|A\|$, where $\rho(A)$ is the spectral radius of the operator A. Thus, we can regard $\mu_\Delta(\cdot)$ as a generalized spectral radius.

Let $\mathcal{K} = H^2(\mathcal{E})$ and $\mathcal{H} = \mathcal{K} \ominus m\mathcal{K}$, with m a finite Blaschke product. Denote

$$I_{H^2} \otimes \Delta := \{I_{H^2} \otimes X : X \in \Delta\}$$
$$I_{H^2} \otimes \Delta' := \{I_{H^2} \otimes X : X \in \Delta'\}$$

Remark 2.2 The subspace $\mathcal{H}$ defined above reduces both $I_{H^2} \otimes \Delta$ and $I_{H^2} \otimes \Delta'$.

Let U be an isometry on $\mathcal{K}$, $\mathcal{H} \subset \mathcal{K}$ invariant for U^* with $T = U_\mathcal{H}$ the compression of U to $\mathcal{H}$ and let

$$\{U_\Delta\}' \;:=\; \{Y : Y \in \{U\}' \cap I_{H^2} \otimes \Delta'\}$$
$$\{T_\Delta\}' \;:=\; \{X : X \in \{T\}' \cap I_{H^2} \otimes \Delta' | \mathcal{H}\}$$

Now we recall some generalizations of the spectral radius [BFT90]. For $A \in \{T\}'$ and $B \in \{U\}'$ we define

$$\rho_T^\Delta(A) \;:=\; \inf\{\|XAX^{-1}\| : X \text{ invertible}, X \in \{T_\Delta\}'\},$$
$$\rho_U^\Delta(B) \;:=\; \inf\{\|YBY^{-1}\| : Y \text{ invertible}, Y \in \{U_\Delta\}'\}.$$

Suppose further that $A\mathcal{H} \subset \mathcal{M}$ where $\mathcal{M}$ is T^* invariant. For A such that $AT = P_\mathcal{M} TA$, we define

$$\rho_{T,\mathcal{M}}^\Delta(A) := \inf\{\|P_{X^*\mathcal{M}} X^{-1} AX\| : X \text{ invertible}, X \in \{T_\Delta\}'\}.$$

For any $X \in \{T_\Delta\}'$, and $\mathcal{M} \subset \mathcal{H}$ invariant for T^*, denote $\mathcal{M}_{X^*} = X^*\mathcal{M}$. Note that $\mathcal{M}_{X^*}$ is also T^* invariant. Moreover, if X is invertible, we have

$$P_{\mathcal{M}_{X^*}} X^{-1} = P_{\mathcal{M}_{X^*}} X^{-1} P_\mathcal{M}, \;\; P_\mathcal{M} X = P_\mathcal{M} X P_{\mathcal{M}_{X^*}}. \qquad (3)$$

The problem we will consider in this paper is the following:

(StLTNP) Structured Left Tangential Nevanlinna–Pick Problem Given the interpolation data $\{(\beta_j; b_j, b_j')\}_1^m$ where $b_j' \in \mathbf{C}^N \setminus \{0\}$, $b_j \in \mathbf{C}^M$ and $\beta_j \in \mathbf{D}$, and a real number $\gamma > 0$. Find necessary and sufficient conditions for the existence of an analytic function $F : \mathbf{D} \to H^\infty(\mathbf{C}^{N\times M})$ such that

i) $b_j'^* F(\beta_j) = b_j^*$, $j = 1, \ldots, m$.

ii) $\|F\|_{\hat{\mu}} < \gamma$.

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where

$$||F||_{\hat{\mu}} := \inf \{ \sup_{z\in\mathbf{D}} \hat{\mu}_\Delta(F(z)) : F \text{ satisfies (i) } \}$$

Remark 2.3 *For simplicity, we will assume that all $\{\beta_j\}_1^m$ are distinct. Also, without loss of generality we will consider $F \in H^\infty(\mathbf{C}^{M\times M})$.*

3 The Structured Commutant Lifting Theorem

Here we state one of the main results of this paper. For the following theorem consider $\mathcal{E}, \Delta, \mathcal{K}, \mathcal{H}, \mathcal{M}, U, T$ as above.

Theorem 3.1 (Structured Left Tangential Commutant-Lifting) *Given $A \in \mathcal{L}(\mathcal{H})$ such that $A\mathcal{H} \subset \mathcal{M}$. If A is such that $AT = P_\mathcal{M} TA$, then there exists $B \in \mathcal{L}(\mathcal{K})$ such that*

$$\rho_{T,\mathcal{M}}^\Delta(A) = \inf\{\rho_U^\Delta(B) : B \in Dil(A)\}$$

where

$$Dil(A) = \{B : BU = UB, P_\mathcal{M} B = AP_\mathcal{H}\}$$

Proof We will generalize the proof in [BFT89b] for the spectral left tangential case. First we will show that $\rho_{T,\mathcal{M}}^\Delta(A) \le \rho_U^\Delta(B)$. Given $\epsilon > 0$, fix $B \in Dil(A)$ such that

$$\rho_U^\Delta(B) = \inf\{\rho_U^\Delta(B_1) : B_1 \in Dil(A)\} + \epsilon$$

By the structured commutant lifting theorem in [BFT90], there exists an invertible $Y \in \{U_\Delta\}'$ such that

$$||Y^{-1}BY|| < \inf\{\rho_U^\Delta(B_1) : B_1 \in Dil(A)\} + \epsilon$$

Since $\mathcal{H}$ is hyperinvariant for U^* (invariant for U^* and all the operators commuting with U^*), the compression of Y, $X = Y_\mathcal{H} = P_\mathcal{H} Y|\mathcal{H}$, is in $\{T_\Delta\}'$ and its inverse is $X^{-1} = (Y^{-1})_\mathcal{H}$. Let $\mathcal{M}_{X^*} = X^*\mathcal{M}$. Using the fact that $B \in Dil(A)$, the reducing property of $\mathcal{H}$ and (3) we have

$$\begin{aligned}
||P_{\mathcal{M}_{X^*}} X^{-1}AX|| &= ||P_{\mathcal{M}_{X^*}} X^{-1}AP_\mathcal{H} X|| \\
&= ||P_{\mathcal{M}_{X^*}} X^{-1}P_\mathcal{M} B|\mathcal{H} X|| \\
&= ||P_{\mathcal{M}_{X^*}} X^{-1}P_\mathcal{H} B|\mathcal{H} X|| \\
&= ||P_{\mathcal{M}_{X^*}} (Y_\mathcal{H}^{-1}) B_\mathcal{H} Y|| \\
&\le ||(Y^{-1}BY)_\mathcal{H}|| \\
&< ||Y^{-1}BY|| + \epsilon \\
\inf_X ||P_{\mathcal{M}_{X^*}} X^{-1}AX|| &< \inf_Y ||Y^{-1}BY|| + \epsilon.
\end{aligned}$$

thus, since $\epsilon > 0$ is arbitrary $\rho_{T,\mathcal{M}}^\Delta(A) \le \rho_U^\Delta(B)$.

To show that $\rho_{T,\mathcal{M}}^\Delta(A) \ge \rho_U^\Delta(B)$, let $\epsilon > 0$ and choose $X \in \{T_\Delta\}'$ invertible such that

$$||P_{\mathcal{M}_{X^*}} X^{-1}AX|| < \rho_{T,\mathcal{M}}^\Delta(A) + \epsilon.$$

From [BFT90] there exists an invertible lifting $Y \in \{U_\Delta\}'$ of X. Setting $A_1 = P_{\mathcal{M}_{X^*}} X^{-1}AX$ we have $A_1 T = T_{\mathcal{M}_{X^*}} A_1$. By the Commutant Lifting Theorem there exists $B_1 \in Dil(A_1)$ such that

$$||A_1|| = ||B_1|| < \rho_{T,\mathcal{M}}^\Delta(A) + \epsilon.$$

Set $B = YB_1Y^{-1}$ and note that

$$\rho_U^\Delta(B) \le ||B_1|| < \rho_{T,\mathcal{M}}^\Delta(A) + \epsilon.$$

It can be verified that $B \in Dil(A) = \{B : BU = UB, P_\mathcal{M} B = AP_\mathcal{H}\}$. Thus

$$\rho_U^\Delta(B) < \rho_{T,\mathcal{M}}^\Delta(A) + \epsilon,$$

and since $\epsilon > 0$ arbitrary we obtain the desired result. □

Remark 3.1 The invariant $\rho_U^\Delta(B)$ is intimately related to $\hat{\mu}_\Delta(B)$. In fact, from corollary 2 in [BFT90] we have that: if $B \in \{U\}'$ is continuous on $\overline{\mathbf{D}}$ then

$$\rho_U^\Delta(B) = \sup_{z\in\mathbf{D}} \hat{\mu}_\Delta(B(z)).$$

4 Structured Nevanlinna–Pick Interpolation

Here we will use Theorem 3 to solve the structured left tangential NP problem. Based on the proof for the spectral tangential NP problem [BFT89b] set the spaces and operators as follows

$$m(z) = \prod_{j=1}^m \frac{z-\beta_j}{1-\overline{\beta}_j z} \quad ; \quad g_{\beta_j}(z) = \frac{1}{1-\overline{\beta}_j z} \quad ; \quad j = 1,\ldots,m$$

$$\mathcal{M} = \sum_{j=1}^m \mathbf{C}\, b_i \otimes g_{\beta_j} \quad ; \quad \mathcal{K} = H^2(\mathbf{C}^M) \quad ; \quad \mathcal{H} = \mathcal{K} \ominus m\mathcal{K}$$

Thus we have $T^*\, v \otimes g_{\beta_j} = \beta_j v \otimes g_{\beta_j}$, $v \in \mathbf{C}^M$.

Define the operator $A : \mathcal{H} \to \mathcal{H}$ by linearity and via its adjoint as

$$A^* b_j' \otimes g_{\beta_j} = b_j \otimes g_{\beta_j}, \quad j = 1,\ldots,m.$$

Then by the tangential commutant lifting theorem all the functions F that interpolate the data $\{(\beta_j; b_j, b_j')\}_1^m$ must be multiplication operators M_F in $Dil(A)$. Now fix $\Delta \in \mathcal{L}(\mathbf{C}^M)$ and consider $D \in \{T_\Delta^*\}'$ invertible and set $\mathcal{M}_D = D\mathcal{M}$. Then necessarily

$$Dv \otimes g_{\beta_j} = D_j v \otimes g_{\beta_j}, \quad D_j, D_j^{-1} \in \Delta', \quad v \in \mathbf{C}^M, \quad j = 1,\ldots,m.$$

Moreover, we have that

$$DA^*(\beta_j; b_j; b_j')D^{-1}|\mathcal{M}_D = A^*(\beta_j; D_j b_j; D_j b_j'),$$

and we conclude that

$$\rho_{T,\mathcal{M}}^\Delta(A) = \inf\{||A(\beta_j; D_j b_j; D_j b_j')|| : D_j, D_j^{-1} \in \Delta'\}.$$

Thus from Theorem 3 and remark 3.1

$$\gamma_{opt} := \rho_{T,\mathcal{M}}^\Delta(A) = \inf\{\sup_{z\in\mathbf{D}} \hat{\mu}_\Delta(B(z)) : B \in Dil(A)\}.$$

It is clear that the spectral tangential problem is solvable if and only if $\gamma \ge \gamma_{opt}$. This fact can be stated in terms of the positivity of certain Hermitian Nevanlinna–Pick matrix, $\mathcal{N}_\gamma$, as the following theorem shows.

Theorem 4.1 (Structured Left Tangential Nevanlinna–Pick) *There exists $F : \mathbf{D} \to H^\infty(\mathbf{C}^{M\times M})$ continuous on $\overline{\mathbf{D}}$ solving StLTNP if and only if there exists invertible matrices $D_j \in \Delta', j = 1,\ldots,m$ such that $\mathcal{N}_\gamma(\beta_j; D_j b_j; D_j b_j') > 0$, where*

$$\mathcal{N}_\gamma(\beta_j; D_j b_j; D_j b_j') := \left[\frac{\gamma^2\langle D_j b_j', D_i b_i'\rangle - \langle D_j b_j, D_i b_i\rangle}{1-\beta_i\overline{\beta}_j}\right]_{1\le i,j\le m}$$

Proof We have that

$$\begin{aligned}
||A(\beta_j; D_j b_j; D_j b_j')|| < \gamma \Leftrightarrow ||A^*(\beta_j; D_j b_j; D_j b_j')|| &< \gamma \\
\Leftrightarrow \gamma^2 I - A(\beta_j; D_j b_j; D_j b_j')A^*(\beta_j; D_j b_j; D_j b_j') &> 0 \\
\Leftrightarrow \mathcal{N}_\gamma(\beta_j; D_j b_j; D_j b_j') &> 0. \quad \square
\end{aligned}$$

Remark 4.1 In this setting, γ_{opt} is the largest γ that makes $\mathcal{N}_\gamma$ singular. Thus we have reduce the existence of a solution to the above problem to a problem in linear algebra.

5 Conclusions

In this paper we have extended our previous work on spectral tangential interpolation to a new class of tangential Nevanlinna–Pick interpolation problems with bounds on the structured singular values of the interpolating functions. The more general case of two-sided spectral and structured interpolation will be reported in [BCFT, Coc93]. Currently we are investigating the computation of some of the invariants appearing in this theory. Our goal is to complete this new structured interpolation theory to develop rigorous analytic procedures for μ-synthesis.

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